Mini Homework 6
Name: - Math 117 - Summer 2022

1) Let V, W be $\mathbb{F}$ vector spaces. Recall that a skew-symmetric map is a function

$$f : V \times V \to W$$

such that $f(v_1, v_2) = -f(v_2, v_1)$

(a) (2 points) Prove that an alternating map is always skew-symmetric

(b) (1 point) Prove that if $char(\mathbb{F}) \neq 2$ then a skew-symmetric map is also alternating (Recall: we discussed char of a field in lecture 1)

Solution:

2) (2 points) Recall that in class we constructed the map $\bigwedge^2(V) \to V \otimes V$ that sends the simple wedge

$$v_1 \wedge v_2 \to v_1 \otimes v_2 - v_2 \otimes v_1$$

Prove that this map is injective.

Remark: In this way, we can think of the exterior product $\bigwedge^2(V) \subseteq V \otimes V$ instead of as a quotient. This is how it is often defined in the literature, when they haven't built up the machinery to talk about quotient spaces. However, this has some major downsides.

1. In our construction, there is a natural map $V \otimes V \to \bigwedge^2(V)$, since it is a quotient space. Viewing the alternating maps as a subspace of the tensor product, there is no natural map back into the subspace (in other words, there isn't a natural way to make a bilinear map into an alternating map).
2. Books get around this by defining a formula to 'alternatize' a bilinear map. However, it is both 1) a very gross formula that involves knowledge of group theory; and 2) depends on certain numbers being divisible in our field. If we only care about $\mathbb{R}$ or $\mathbb{C}$ then this latter issue is okay, but many many situations naturally arise where we have to discuss more general fields.
3. This is all to say, once we took our medicine, and learned quotient spaces, our life really does become better, and we can do more powerful constructions. Anyway, on-wards and upwards ...

Solution:

3) Prove that

(a) (2 points) $v_1 \wedge v_2 = 0$ in $\bigwedge^2(V)$ iff every alternating bilinear map

$$f : V \times V \to W$$

has $f(v_1, v_2) = 0$

(b) (1 point) Conclude that $\bigwedge^2(V) = 0 \iff$ every symmetric bilinear map $f : V \times V \to W$ is identically 0 (ie, $f(v_1, v_2) = 0$ for all $v_1, v_2, \in V$

(c) (2 points) Prove that $v_1 \wedge v_2 = v_1' \wedge v_2'$ in $\bigwedge^2(V)$ iff every alternating bilinear map

$$f : V \times V \to W$$

has the property that $f(v_1, v_2) = f(v_1', v_2')$

Solution: